

MORE PROPERTIES OF YETTER-DRINFELD MODULES OVER QUASI-HOPF ALGEBRAS

D. BULACU, S. CAENEPEEL, AND F. PANAITTE

ABSTRACT. We generalize various properties of Yetter-Drinfeld modules over Hopf algebras to quasi-Hopf algebras. The dual of a finite dimensional Yetter-Drinfeld module is again a Yetter-Drinfeld module. The algebra H_0 in the category of Yetter-Drinfeld modules that can be obtained by modifying the multiplication in a proper way is quantum commutative. We give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules, and deduce the existence and uniqueness of integrals from it.

1. INTRODUCTION

The motivation for studying Yetter-Drinfeld modules over quasi-Hopf algebras is the same as for Hopf algebras. It is well known that for any finite dimensional Hopf algebra H the category of Yetter-Drinfeld modules ${}_H\mathcal{YD}^H$ is isomorphic to the category of modules over the quantum double $D(H)$. From a categorical point of view, the quantum double $D(H)$ arises by considering the center $\mathcal{Z}({}_H\mathcal{M})$ of the monoidal category ${}_H\mathcal{M}$ of left H -modules. More precisely, one has $\mathcal{Z}({}_H\mathcal{M}) \simeq {}_{D(H)}\mathcal{M}$ if H is finite dimensional. Actually, the category of Yetter-Drinfeld modules appears as an intermediate step in the proof of this isomorphism: one first proves that $\mathcal{Z}({}_H\mathcal{M}) \simeq {}_H\mathcal{YD}^H$, and then ${}_H\mathcal{YD}^H \simeq {}_{D(H)}\mathcal{M}$, where the finite dimensionality is not needed in the proof of the first isomorphism, see [16] for full detail.

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [13]; a categorical interpretation is the following: a quasi-bialgebra H is an algebra with the additional structure that is needed to make the category of left H -modules, with the tensor product over k as tensor product and k as unit object into a monoidal category. The difference with a usual bialgebra is that we do not require that the associativity isomorphism coincides with the associativity in the category of vector spaces. A quasi-Hopf algebra is a quasi-bialgebra with additional structure making the category of finite dimensional H -modules into a monoidal category with duality. The center construction $\mathcal{Z}(\mathcal{C})$ can be applied to any monoidal category \mathcal{C} . Majid [19] computed the center of the category of left modules over a quasi-Hopf algebra H , and introduced the category of Yetter-Drinfeld modules over H . Hausser and Nill [14], [15] constructed the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra H , and proved that ${}_H\mathcal{YD}^H \simeq {}_{D(H)}\mathcal{M}$. Recently, Schauenburg [22]

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gave the equivalence between the category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ and the category ${}^H_H\mathcal{M}_H^H$ of Hopf bimodules. In [5], the relation between Yetter-Drinfeld modules and Radford's biproduct is studied. In [4], the rigidity of the category of Yetter-Drinfeld modules is investigated, as well as the relations between left, left-right, right-left and right Yetter-Drinfeld modules.

In this paper, which can be seen as a sequel to [4], we continue our investigations of properties of Yetter-Drinfeld modules. In Section 3, we show that the linear dual of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module.

It was shown in [7], [5] that the multiplication on H can be modified in such a way that we obtain an algebra in the category of left Yetter-Drinfeld modules. The main result of Section 4 is that H_0 is quantum commutative.

In Section 5, we will generalize Doi's results [12] about Hopf modules in the category of Yetter-Drinfeld modules to our situation: we give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules over a quasi-Hopf algebras, and we use this result to obtain the existence and uniqueness of integrals for a finite dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$. We apply this to the braided Hopf algebra considered in Section 4, in the case where H is finite dimensional and quasitriangular.

2. PRELIMINARY RESULTS

2.1. Quasi-Hopf algebras. We work over a commutative field k . All algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . Following Drinfeld [13], a quasi-bialgebra is a fourtuple $(H, \Delta, \varepsilon, \Phi)$, where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \rightarrow H \otimes H$ and $\varepsilon : H \rightarrow k$ are algebra homomorphisms satisfying the identities

$$\begin{aligned} (1) \quad & (id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1}, \\ (2) \quad & (id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h, \end{aligned}$$

for all $h \in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$\begin{aligned} (3) \quad & (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi), \\ (4) \quad & (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1. \end{aligned}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras [23] we use the notation $\Delta(h) = \sum h_1 \otimes h_2$. Since Δ is only quasi-coassociative we adopt the further notation

$$(\Delta \otimes id)(\Delta(h)) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\begin{aligned} \Phi &= \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \dots \\ \Phi^{-1} &= \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 = \dots \end{aligned}$$

A quasi-bialgebra H is called a quasi-Hopf algebra if there exists an anti-automorphism S of the algebra H and $\alpha, \beta \in H$ such that:

$$(5) \quad \sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta,$$

$$(6) \quad \sum X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2\beta S(x^3) = 1,$$

for all $h \in H$. It is shown in [9] that the condition that the antipode is bijective follows automatically from the other axioms in the case where H is finite dimensional. Observe that the antipode of a quasi-Hopf algebra is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β , we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. The identities (2-4) also imply that

$$(7) \quad (\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1.$$

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have H^{op} , H^{cop} and $H^{\text{op,cop}}$ as quasi-Hopf algebras, where “op” means opposite multiplication and “cop” means opposite comultiplication. The reassociators of these three quasi-Hopf algebras are $\Phi_{\text{op}} = \Phi^{-1}$, $\Phi_{\text{cop}} = (\Phi^{-1})^{321}$, $\Phi_{\text{op,cop}} = \Phi^{321}$, the antipodes are $S_{\text{op}} = S_{\text{cop}} = (S_{\text{op,cop}})^{-1} = S^{-1}$, and the elements α, β are $\alpha_{\text{op}} = S^{-1}(\beta)$, $\beta_{\text{op}} = S^{-1}(\alpha)$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$, $\beta_{\text{cop}} = S^{-1}(\beta)$, $\alpha_{\text{op,cop}} = \beta$ and $\beta_{\text{op,cop}} = \alpha$.

Recall next that the definition of a quasi-Hopf algebra is “twist coinvariant”, in the following sense. An invertible element $F \in H \otimes H$ is called a *gauge transformation* or *twist* if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, antipode and the elements α and β by

$$(8) \quad \Delta_F(h) = F\Delta(h)F^{-1},$$

$$(9) \quad \Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1),$$

$$(10) \quad \alpha_F = \sum S(G^1)\alpha G^2, \quad \beta_F = \sum F^1\beta S(F^2).$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. The corresponding statement for a quasi-Hopf algebra is the following: there exists a gauge transformation $f \in H \otimes H$ such that

$$(11) \quad f\Delta(S(h))f^{-1} = \sum (S \otimes S)(\Delta^{\text{cop}}(h)),$$

for all $h \in H$, where $\Delta^{\text{cop}}(h) = \sum h_2 \otimes h_1$. The element f can be computed explicitly. First set

$$(12) \quad \sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}),$$

$$(13) \quad \sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$(14) \quad \gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = \sum B^1\beta S(B^4) \otimes B^2\beta S(B^3).$$

Then f and f^{-1} are given by the formulas

$$(15) \quad f = \sum (S \otimes S)(\Delta^{\text{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)),$$

$$(16) \quad f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{op}}(x^3)).$$

Moreover, f satisfies the following relations:

$$(17) \quad f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta.$$

Furthermore the corresponding twisted reassociator (see (9)) is given by

$$(18) \quad \Phi_f = \sum (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1).$$

In a Hopf algebra H , we obviously have the identity

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \text{ for all } h \in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [14, 15], we define

$$(19) \quad p_R = \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3),$$

$$(20) \quad q_R = \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2,$$

$$(21) \quad p_L = \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3,$$

$$(22) \quad q_L = \sum q_L^1 \otimes q_L^2 = \sum S(x^1) \alpha x^2 \otimes x^3.$$

We then have, for all $h \in H$,

$$(23) \quad \sum \Delta(h_1) p_R [1 \otimes S(h_2)] = p_R(h \otimes 1),$$

$$(24) \quad \sum [1 \otimes S^{-1}(h_2)] q_R \Delta(h_1) = (h \otimes 1) q_R,$$

$$(25) \quad \sum \Delta(h_2) p_L [S^{-1}(h_1) \otimes 1] = p_L(1 \otimes h),$$

$$(26) \quad \sum [S(h_1) \otimes 1] q_L \Delta(h_2) = (1 \otimes h) q_L,$$

and

$$(27) \quad (q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} = \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)] \\ [1 \otimes S^{-1}(f^2) \otimes S^{-1}(f^1)](id \otimes \Delta)(q_R \Delta(X^1)),$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (15).

A quasi-Hopf algebra H is quasitriangular if there exists an element $R \in H \otimes H$ such that

$$(28) \quad (\Delta \otimes id)(R) = \sum \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi,$$

$$(29) \quad (id \otimes \Delta)(R) = \sum \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1},$$

$$(30) \quad \Delta^{\text{cop}}(h)R = R\Delta(h), \text{ for all } h \in H,$$

$$(31) \quad (\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1.$$

Here we used the following notation: if σ is a permutation of $\{1, 2, 3\}$, then we write $\Phi_{\sigma(1)\sigma(2)\sigma(3)} = \sum X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}$; R_{ij} means R acting non-trivially

on the i -th and j -th tensor factors of $H \otimes H \otimes H$.

It is shown in [10] that R is invertible. Furthermore, the element

$$(32) \quad u = \sum S(R^2 p^2) \alpha R^1 p^1,$$

with $p_R = \sum p^1 \otimes p^2$ defined as in (19), is invertible in H , and

$$(33) \quad u^{-1} = \sum X^1 R^2 p^2 S(S(X^2 R^1 p^1) \alpha X^3),$$

$$(34) \quad \varepsilon(u) = 1 \text{ and } S^2(h) = uhu^{-1},$$

for all $h \in H$. Consequently the antipode S is bijective, so, as in the Hopf algebra case, the assumptions about invertibility of R and bijectivity of S can be dropped. Moreover, the R -matrix $R = \sum R^1 \otimes R^2$ satisfies the identity (see [1], [15], [10]):

$$(35) \quad f_{21} R f^{-1} = (S \otimes S)(R)$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (15), and $f_{21} = \sum f^2 \otimes f^1$.

2.2. Monoidal categories. A monoidal or tensor category is a sextuple $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$, where \mathcal{C} is a category, \otimes is a functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (called the tensor product), $\underline{1}$ is an object of \mathcal{C} , and

$$\begin{aligned} a_{U,V,W} : (U \otimes V) \otimes W &\rightarrow U \otimes (V \otimes W) \\ l_V : V \otimes \underline{1} &\cong V \\ r_V : \underline{1} \otimes V &\cong V \end{aligned}$$

are natural isomorphisms satisfying certain coherence conditions, see for example [16, 18, 20]. An object V of a monoidal category \mathcal{C} has a left dual if there exists an object V^* and morphisms $\text{ev}_V : V^* \otimes V \rightarrow \underline{1}$, $\text{coev}_V : \underline{1} \rightarrow V \otimes V^*$ in \mathcal{C} such that

$$(36) \quad l_V^{-1} \circ (\text{id}_V \otimes \text{ev}_V) \circ a_{V,V^*,V} \circ (\text{coev}_V \otimes \text{id}_V) \circ r_V = \text{id}_V,$$

$$(37) \quad r_{V^*}^{-1} \circ (\text{ev}_V \otimes \text{id}_{V^*}) \circ a_{V^*,V,V^*}^{-1} \circ (\text{id}_{V^*} \otimes \text{coev}_V) \circ l_{V^*} = \text{id}_{V^*}.$$

\mathcal{C} is called a rigid monoidal category if every object of \mathcal{C} has a dual.

A braided monoidal category is a monoidal category equipped with a commutativity natural isomorphism $c_{U,V} : U \otimes V \rightarrow V \otimes U$, compatible with the unit and the associativity.

In a braided monoidal category, we can define algebras, coalgebras, bialgebras and Hopf algebras. For example, a bialgebra $(B, \underline{m}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon})$ consists of $B \in \mathcal{C}$, a multiplication $\underline{m} : B \otimes B \rightarrow B$ which is associative up to the natural isomorphism a , and a unit $\underline{\eta} : \underline{1} \rightarrow B$ such that $\underline{m} \circ (\underline{\eta} \otimes \text{id}) = \underline{m} \circ (\text{id} \otimes \underline{\eta}) = \text{id}$. The properties of the comultiplication $\underline{\Delta}$ and the counit $\underline{\varepsilon}$ are similar. In addition, $\underline{\Delta} : B \rightarrow B \otimes B$ has to be an algebra morphism, where $B \otimes B$ is an algebra with multiplication $\underline{m}_{B \otimes B}$, defined as the composition

$$(38) \quad \begin{array}{ccc} (B \otimes B) \otimes (B \otimes B) & \xrightarrow{a} & B \otimes (B \otimes (B \otimes B)) \\ & \xrightarrow{\text{id} \otimes a^{-1}} & B \otimes ((B \otimes B) \otimes B) \\ & \xrightarrow{\text{id} \otimes c \otimes \text{id}} & B \otimes ((B \otimes B) \otimes B) \\ & \xrightarrow{\text{id} \otimes a} & B \otimes (B \otimes (B \otimes B)) \\ & \xrightarrow{a^{-1}} & (B \otimes B) \otimes (B \otimes B) \\ & \xrightarrow{\underline{m} \otimes \underline{m}} & B \otimes B \end{array}$$

A Hopf algebra B is a bialgebra with a morphism $\underline{S} : B \rightarrow B$ in \mathcal{C} (the antipode) satisfying the usual axioms $\underline{m} \circ (\underline{S} \otimes \text{id}) \circ \underline{\Delta} = \underline{\eta} \circ \underline{\varepsilon} = \underline{m} \circ (\text{id} \otimes \underline{S}) \circ \underline{\Delta}$. It is known, see

e.g. [21], that the antipode \underline{S} of a Hopf algebra B in a braided monoidal category \mathcal{C} is an antialgebra and anticoalgebra morphism, in the sense that

$$(39) \quad \underline{S} \circ \underline{m} = \underline{m} \circ (\underline{S} \otimes \underline{S}) \circ c_{B,B} \text{ and } \underline{\Delta} \circ \underline{S} = c_{B,B} \circ (\underline{S} \otimes \underline{S}) \circ \underline{\Delta}.$$

Recall also that an algebra A in a braided monoidal category \mathcal{C} is called quantum commutative if $\underline{m} \circ c_{A,A} = \underline{m}$.

Assume that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra, and let U, V, W be left H -modules. We define a left H -action on $U \otimes V$ by

$$h \cdot (u \otimes v) = \sum h_1 \cdot u \otimes h_2 \cdot v.$$

We have isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ in ${}_H\mathcal{M}$ given by

$$(40) \quad a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)).$$

The counit $\varepsilon : H \rightarrow k$ makes $k \in {}_H\mathcal{M}$, and the natural isomorphisms $\lambda : k \otimes H \rightarrow H$ and $\rho : H \otimes k \rightarrow H$ are in ${}_H\mathcal{M}$. With this structures, $({}_H\mathcal{M}, \otimes, k, a, \lambda, \rho)$ is a monoidal category.

If H is a quasi-Hopf algebra then the category of finite dimensional left H -modules is rigid; the left dual of V is V^* with the H -module structure given by $(h \cdot \varphi)(v) = \varphi(S(h) \cdot v)$, for all $v \in V$, $\varphi \in V^*$, $h \in H$ and with

$$(41) \quad \text{ev}_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad \text{coev}_V(1) = \sum_{i=1}^n \beta \cdot v_i \otimes v^i,$$

where $\{v_i\}$ is a basis in V with dual basis $\{v^i\}$.

Now let H be a quasitriangular quasi-Hopf algebra, with R -matrix $R = \sum R^1 \otimes R^2$. For two left H -modules U and V , we define

$$c_{U,V} : U \otimes V \rightarrow V \otimes U$$

by

$$(42) \quad c_{U,V}(u \otimes v) = \sum R^2 \cdot v \otimes R^1 \cdot u$$

and then $({}_H\mathcal{M}, \otimes, k, a, \lambda, \rho, c)$ is a braided monoidal category (cf. [16] or [20]).

3. YETTER-DRINFELD MODULES AND THE QUASI-YANG-BAXTER EQUATION

From [19], we recall the notion of Yetter-Drinfeld module over a quasi-bialgebra.

Definition 3.1. Let H be a quasi-bialgebra with reassociator Φ . A left H -module M together with a left H -coaction

$$\lambda_M : M \rightarrow H \otimes M, \quad \lambda_M(m) = \sum m_{(-1)} \otimes m_{(0)}$$

is called a left Yetter-Drinfeld module if the following equalities hold, for all $h \in H$ and $m \in M$:

$$(43) \quad \begin{aligned} & \sum X^1 m_{(-1)} \otimes (X^2 \cdot m_{(0)})_{(-1)} X^3 \otimes (X^2 \cdot m_{(0)})_{(0)} \\ &= \sum X^1 (Y^1 \cdot m)_{(-1)_1} Y^2 \otimes X^2 (Y^1 \cdot m)_{(-1)_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{(0)} \end{aligned}$$

$$(44) \quad \sum \varepsilon(m_{(-1)}) m_{(0)} = m$$

$$(45) \quad \sum h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = \sum (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}.$$

The category of left Yetter-Drinfeld H -modules and k -linear maps that intertwine the H -action and H -coaction is denoted by ${}^H_H\mathcal{YD}$. In [19] it is shown that ${}^H_H\mathcal{YD}$ is a prebraided monoidal category. The forgetful functor ${}^H_H\mathcal{YD} \rightarrow {}_H\mathcal{M}$ is monoidal, and the coaction on the tensor product $M \otimes N$ of two Yetter-Drinfeld modules M and N is given by

$$(46) \quad \lambda_{M \otimes N}(m \otimes n) = \sum X^1(x^1 Y^1 \cdot m)_{(-1)} x^2 (Y^2 \cdot n)_{(-1)} Y^3$$

$$(47) \quad \otimes X^2 \cdot (x^1 Y^1 \cdot m)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{(0)}.$$

The braiding is given by

$$(48) \quad c_{M,N}(m \otimes n) = \sum m_{(-1)} \cdot n \otimes m_{(0)}.$$

This braiding is invertible if H is a quasi-Hopf algebra [5], and its inverse is then given by

$$(49) \quad c_{M,N}^{-1}(n \otimes m) = \sum y_1^3 X^2 \cdot (x^1 \cdot m)_{(0)} \otimes S^{-1}(S(y^1) \alpha y^2 X^1 (x^1 \cdot m)_{(-1)} x^2 \beta S(y_2^3 X^3 x^3)) \cdot n.$$

Let (H, R) be a quasitriangular quasi-bialgebra. It is well-known (see for example [16]) that R satisfies the so-called quasi-Yang-Baxter equation in $H \otimes H \otimes H$:

$$R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi = \Phi_{321} R_{23} \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12}.$$

On the other hand, if H is a bialgebra and M is a left-right Yetter-Drinfeld module over H , with structures

$$\begin{aligned} H \otimes M &\rightarrow M, & h \otimes m &\mapsto h \cdot m; \\ M &\rightarrow M \otimes H, & m &\mapsto \sum m_{(0)} \otimes m_{(1)}, \end{aligned}$$

then the map $R_M : M \otimes M \rightarrow M \otimes M$, $R_M(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$ is a solution in $\text{End}(M \otimes M \otimes M)$ of the quantum Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

see for instance [17].

We will show a similar result for quasi-bialgebras; first we define left-right Yetter-Drinfeld modules over quasi-bialgebras as follows

$${}_H\mathcal{YD}^H = {}_{H^{\text{cop}}}^H\mathcal{YD}.$$

This is stated more explicitly in the next definition.

Definition 3.2. Let H be a quasi-bialgebra. A k -linear space M with a left H -action $h \otimes m \mapsto h \cdot m$, and a right H -coaction $M \rightarrow M \otimes H$, $m \mapsto \sum m_{(0)} \otimes m_{(1)}$ is called a left-right Yetter-Drinfeld module if the following relations hold, for all $m \in M$ and $h \in H$:

$$(50) \quad \begin{aligned} &\sum (x^2 \cdot m_{(0)})_{(0)} \otimes (x^2 \cdot m_{(0)})_{(1)} x^1 \otimes x^3 m_{(1)} \\ &= \sum x^1 \cdot (y^3 \cdot m)_{(0)} \otimes x^2 (y^3 \cdot m)_{(1)_1} y^1 \otimes x^3 (y^3 \cdot m)_{(1)_2} y^2 \end{aligned}$$

$$(51) \quad \sum \varepsilon(m_{(1)}) m_{(0)} = m$$

$$(52) \quad \sum h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = \sum (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1.$$

Proposition 3.3. *Let H be a quasi-bialgebra and $M \in {}_H\mathcal{YD}^H$. The map $R = R_M : M \otimes M \rightarrow M \otimes M$, $R(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$, is a solution of the quasi-Yang-Baxter equation*

$$(53) \quad R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi = \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}$$

on $\text{End}(M \otimes M \otimes M)$.

We considered R_{12}, Φ_{312} , etc. as elements in $\text{End}(M \otimes M \otimes M)$ by left multiplication, for example $R_{12}(l \otimes m \otimes n) = \sum R^1 \cdot l \otimes R^2 \cdot m \otimes n$, $\Phi_{312}(l \otimes m \otimes n) = \sum X^2 \cdot l \otimes X^3 \cdot m \otimes X^1 \cdot n$ etc.

Proof. ${}_H\mathcal{YD}^H$ is a prebraided category, hence the result is a consequence of the fact (see [16]) that the braiding satisfies the categorical version of the Yang-Baxter equation. A direct proof is also possible. For all $l, m, n \in M$, we compute that

$$\begin{aligned} & R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi(l \otimes m \otimes n) \\ &= \sum (Y^3x^3(X^3 \cdot n)_{(1)}X^2 \cdot m)_{(1)}Y^2(x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)}x^1X^1 \cdot l \\ & \quad \otimes (Y^3x^3(X^3 \cdot n)_{(1)}X^2 \cdot m)_{(0)} \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)} \\ (50) \quad &= \sum (Y^3x^3(y^3X^3 \cdot n)_{(1)_2}y^2X^2 \cdot m)_{(1)}Y^2x^2(y^3X^3 \cdot n)_{(1)_1}y^1X^1 \cdot l \\ & \quad \otimes (Y^3x^3(y^3X^3 \cdot n)_{(1)_2}y^2X^2 \cdot m)_{(0)} \otimes Y^1x^1 \cdot (y^3X^3 \cdot n)_{(0)} \\ &= \sum (n_{(1)_2} \cdot m)_{(1)}n_{(1)_1} \cdot l \otimes (n_{(1)_2} \cdot m)_{(0)} \otimes n_{(0)} \\ (52) \quad &= \sum n_{(1)_2}m_{(1)} \cdot l \otimes n_{(1)_1} \cdot m_{(0)} \otimes n_{(0)} \end{aligned}$$

and

$$\begin{aligned} & \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}(l \otimes m \otimes n) \\ &= \sum Y^3x^3(X^3 \cdot n)_{(1)}X^2m_{(1)} \cdot l \otimes Y^2(x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)}x^1X^1 \cdot m_{(0)} \\ & \quad \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)} \\ (50) \quad &= \sum Y^3x^3(y^3X^3 \cdot n)_{(1)_2}y^2X^2m_{(1)} \cdot l \otimes Y^2x^2(y^3X^3 \cdot n)_{(1)_1}y^1X^1 \cdot m_{(0)} \\ & \quad \otimes Y^1x^1 \cdot (y^3X^3 \cdot n)_{(0)} \\ &= \sum n_{(1)_2}m_{(1)} \cdot l \otimes n_{(1)_1} \cdot m_{(0)} \otimes n_{(0)} \end{aligned}$$

and (53) follows. \square

We will now present a generalization of [17, Prop. 4.4.2], stating that the dual M^* of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module and that $R_{M^*} = R_M^*$.

First we define right-left Yetter-Drinfeld modules for quasi-bialgebras as follows:

$${}_H\mathcal{YD}_H = {}_{H^{\text{op}}, \text{cop}}\mathcal{YD}^{H^{\text{op}}, \text{cop}}.$$

More explicitly:

Definition 3.4. Let H be a quasi-bialgebra. A k -linear space M with a right H -action $m \otimes h \mapsto m \cdot h$, and a left H -coaction $M \rightarrow H \otimes M$, $m \mapsto \sum m_{(-1)} \otimes m_{(0)}$ is called a right-left Yetter-Drinfeld module if the following relations hold, for all

$m \in M$ and $h \in H$:

$$(54) \quad \begin{aligned} & \sum m_{(-1)}x^1 \otimes x^3(m_{(0)} \cdot x^2)_{(-1)} \otimes (m_{(0)} \cdot x^2)_{(0)} \\ &= \sum y^2(m \cdot y^1)_{(-1)_1}x^1 \otimes y^3(m \cdot y^1)_{(-1)_2}x^2 \otimes (m \cdot y^1)_{(0)} \cdot x^3 \end{aligned}$$

$$(55) \quad \sum \varepsilon(m_{(-1)})m_{(0)} = m$$

$$(56) \quad \sum m_{(-1)}h_1 \otimes m_{(0)} \cdot h_2 = \sum h_2(m \cdot h_1)_{(-1)} \otimes (m \cdot h_1)_{(0)}.$$

For $M \in {}^H\mathcal{YD}_H$, we consider the map

$$R_M : M \otimes M \rightarrow M \otimes M, \quad R_M(m \otimes n) = \sum m \cdot n_{(-1)} \otimes n_{(0)}.$$

If we consider M as an object in ${}_{H^{op, cop}}\mathcal{YD}^{H^{op, cop}}$, then we obtain the same map R_M , so R_M is also a solution of the corresponding quasi-Yang-Baxter equation, which is obtained after replacing Φ by $\Phi_{op, cop} = \Phi^{321}$.

Now let M be a finite dimensional right-left Yetter-Drinfeld module. Then M^* is a left H -module, with action given by $(h \cdot m^*)(m) = m^*(m \cdot h)$, for all $h \in H, m \in M, m^* \in M^*$. We also define a k -linear map $M^* \rightarrow M^* \otimes H$, $m^* \mapsto \sum m^*_{(0)} \otimes m^*_{(1)}$, by the condition

$$(57) \quad \sum m^*_{(0)}(m)m^*_{(1)} = \sum m^*(m_{(0)})m_{(-1)}$$

for all $m \in M$. We can prove now the following result.

Proposition 3.5. *Let H be a quasi-bialgebra, M a finite dimensional right-left Yetter-Drinfeld module. Then*

- (i) $M^* \in {}_H\mathcal{YD}^H$;
- (ii) $R_{M^*} = R_M^*$.

Proof. (i) We prove that (50), (51), (52) are satisfied. For $m^* \in M^*$ and $m \in M$, we compute:

$$\begin{aligned} & \sum (x^2 \cdot m^*_{(0)})_{(0)}(m)(x^2 \cdot m^*_{(0)})_{(1)}x^1 \otimes x^3m^*_{(1)} \\ (57) \quad &= \sum (x^2 \cdot m^*_{(0)})_{(0)}(m_{(0)})m_{(-1)}x^1 \otimes x^3m^*_{(1)} \\ &= \sum m^*_{(0)}(m_{(0)} \cdot x^2)m_{(-1)}x^1 \otimes x^3m^*_{(1)} \\ (54) \quad &= \sum m^*((m \cdot y^1)_{(0)} \cdot x^3)y^2(m \cdot y^1)_{(-1)_1}x^1 \otimes y^3(m \cdot y^1)_{(-1)_2}x^2 \\ &= \sum (x^3 \cdot m^*)((m \cdot y^1)_{(0)})y^2(m \cdot y^1)_{(-1)_1}x^1 \otimes y^3(m \cdot y^1)_{(-1)_2}x^2 \\ (57) \quad &= \sum (x^3 \cdot m^*)_{(0)}(m \cdot y^1)y^2(x^3 \cdot m^*)_{(1)_1}x^1 \otimes y^3(x^3 \cdot m^*)_{(1)_2}x^2 \\ &= \sum (y^1 \cdot (x^3 \cdot m^*)_{(0)})(m)y^2(x^3 \cdot m^*)_{(1)_1}x^1 \otimes y^3(x^3 \cdot m^*)_{(1)_2}x^2 \end{aligned}$$

so obtain (50). Now we compute:

$$\begin{aligned} & \sum \varepsilon(m^*_{(1)})m^*_{(0)}(m) = \sum \varepsilon(m^*_{(0)}(m)m^*_{(1)}) \\ (57) \quad &= \sum \varepsilon(m^*(m_{(0)})m_{(-1)}) = \sum m^*(\varepsilon(m_{(-1)})m_{(0)}) = m^*(m), \end{aligned}$$

using (55) at the last step. Thus (51) holds. For $h \in H$, we compute:

$$\begin{aligned}
 & \sum (h_1 \cdot m_{(0)}^*)(m) h_2 m_{(1)}^* = \sum m_{(0)}^*(m \cdot h_1) h_2 m_{(1)}^* \\
 (57) \quad &= \sum m^*((m \cdot h_1)_{(0)}) h_2 (m \cdot h_1)_{(-1)} \\
 (56) \quad &= \sum m^*(m_{(0)} \cdot h_2) m_{(-1)} h_1 \\
 &= \sum (h_2 \cdot m^*)(m_{(0)}) m_{(-1)} h_1 \\
 &= \sum (h_2 \cdot m^*)_{(0)}(m) (h_2 \cdot m^*)_{(1)} h_1
 \end{aligned}$$

and (52) follows.

(ii) We identify $(M \otimes M)^* = M^* \otimes M^*$, and we prove that R_{M^*} and R_M^* coincide as maps $M^* \otimes M^* \rightarrow M^* \otimes M^*$. For $m, n \in M$ and $m^*, n^* \in M^*$, we compute:

$$\begin{aligned}
 R_{M^*}(m^* \otimes n^*)(m \otimes n) &= \sum (n_{(1)}^* \cdot m^*)(m) n_{(0)}^*(n) \\
 &= \sum m^*(m \cdot n_{(1)}^*) n_{(0)}^*(n) \\
 (57) \quad &= \sum m^*(m \cdot n_{(-1)}) n^*(n_{(0)}) \\
 &= (m^* \otimes n^*)(R_M(m \otimes n)) \\
 &= R_M^*(m^* \otimes n^*)(m \otimes n),
 \end{aligned}$$

as needed. \square

4. THE QUANTUM COMMUTATIVITY OF H_0

Let H be a Hopf algebra. It is well-known that H is an algebra in the monoidal category ${}^H_H\mathcal{YD}$, with left action and coaction given by

$$h \triangleright h' = \sum h_1 h' S(h_2), \quad \lambda(h) = \sum h_1 \otimes h_2.$$

Moreover, H is quantum commutative as an algebra in ${}^H_H\mathcal{YD}$, see for example [11]. We will now prove a similar result for quasi-Hopf algebras. Let H be a quasi-Hopf algebra. In [7], a new multiplication on H was introduced; this multiplication is given by the formula

$$(58) \quad h \circ h' = \sum X^1 h S(x^1 X^2) \alpha x^2 X_1^3 h' S(x^3 X_2^3)$$

for all $h, h' \in H$. β is a unit for this multiplication \circ . Let H_0 be the k -linear space H , with multiplication \circ , and left H -action given by

$$(59) \quad h \triangleright h' = \sum h_1 h' S(h_2).$$

Then H_0 is a left H -module algebra. In H_0 , we also define a left H -coaction, as follows

$$\begin{aligned}
 \lambda_{H_0}(h) &= \sum h_{(-1)} \otimes h_{(0)} \\
 (60) \quad &= \sum X^1 Y_1^1 h_1 g^1 S(q^2 Y_2^2) Y^3 \otimes X^2 Y_2^1 h_2 g^2 S(X^3 q^1 Y_1^2),
 \end{aligned}$$

where $f^{-1} = \sum g^1 \otimes g^2$ and $q_R = \sum q^1 \otimes q^2$ are the elements defined by (16) and (19). Then H_0 is an algebra in ${}^H_H\mathcal{YD}$, see [5] for details. In Proposition 4.2, we will show that H_0 is quantum commutative. But first we need the following formulas, which are of independent interest. Recall that $q_R = \sum q^1 \otimes q^2$, $q_L = \sum f^1 \otimes f^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are defined by (20), (22), (15) and (16).

Lemma 4.1. *Let H be a quasi-Hopf algebra. Then we have*

$$(61) \quad \sum q^1 y^1 \otimes S(q^2 y^2) y^3 = 1 \otimes \alpha,$$

$$(62) \quad \Phi(\Delta \otimes id)(f^{-1}) = \sum g^1 S(X^3) f^1 \otimes g_1^2 G^1 S(X^2) f^2 \otimes g_2^2 G^2 S(X^1),$$

$$(63) \quad \sum S(g^1) \alpha g^2 = S(\beta), \quad \sum f^1 \beta S(f^2) = S(\alpha),$$

$$(64) \quad \sum S(q_2^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2) = (id \otimes S)(q_L),$$

Proof. (61) and (62) are a direct consequence of (19) and (18). (63) has been proved in [6, Lemma 2.6] and [10, Lemma 2.5]. We are left to prove (64). Using (27), we obtain:

$$(id \otimes \Delta)(q) = \sum (1 \otimes S^{-1}(x^3 g^2) \otimes S^{-1}(x^2 g^1))(q \otimes 1)(\Delta \otimes id)(q) \Phi^{-1}(id \otimes \Delta)(\Delta(x^1))$$

and, using the formula (see [8])

$$(\Delta \otimes id)(q) \Phi^{-1} = \sum Y^1 \otimes q^1 Y_1^2 \otimes S^{-1}(Y^3) q^2 Y_2^2,$$

we obtain

$$(65) \quad (id \otimes \Delta)(q) = \sum Q^1 Y^1 x_1^1 \otimes S^{-1}(x^3 g^2) Q^2 q^1 Y_1^2 x_{(2,1)}^1 \otimes S^{-1}(Y^3 x^2 g^1) q^2 Y_2^2 x_{(2,2)}^1$$

where $q_R = \sum q^1 \otimes q^2 = \sum Q^1 \otimes Q^2$. Now we compute

$$\begin{aligned} & \sum S(q_2^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2) \\ (65) &= \sum S(q^2 Y_2^2 x_{(2,2)}^1 X^3) Y^3 x^2 \otimes S(Q^1 Y^1 x_1^1 X^1 \beta S(Q^2 q^1 Y_1^2 x_{(2,1)}^1 X^2) x^3) \\ (1) &= \sum S(q^2 Y_2^2 X^3 x_2^1) Y^3 x^2 \otimes S(Q^1 Y^1 X^1 x_{(1,1)}^1 \beta S(Q^2 q^1 Y_1^2 X^2 x_{(1,2)}^1) x^3) \\ (5) &= \sum S(q^2 Y_2^2 X^3 x^1) Y^3 x^2 \otimes S(Q^1 Y^1 X^1 \beta S(Q^2 q^1 Y_1^2 X^2) x^3) \\ (3) &= \sum S(q^2 y^2 X_1^3 Y^2 x^1) y^3 X_2^3 Y^3 x^2 \otimes S(Q^1 X^1 Y_1^1 \beta S(Q^2 q^1 y^1 X^2 Y_2^1) x^3) \\ (5, 61) &= \sum S(X_1^3 x^1) \alpha X_2^3 x^2 \otimes S(Q^1 X^1 \beta S(Q^2 X^2) x^3) \\ (5, 7) &= \sum S(x^1) \alpha x^2 \otimes S(Q^1 \beta S(Q^2) x^3) \\ (19, 6) &= \sum S(x^1) \alpha x^2 \otimes S(x^3), \end{aligned}$$

as needed. \square

We can prove now the main result of this Section.

Proposition 4.2. *Let H be a quasi-Hopf algebra. Then H_0 is quantum commutative as an algebra in ${}^H_H \mathcal{YD}$, that is, for all $h, h' \in H$:*

$$h \circ h' = \sum (h_{(-1)} \triangleright h') \circ h_{(0)}.$$

Proof. For all $h, h' \in H$ we compute:

$$\begin{aligned}
& \sum (h_{(-1)} \triangleright h') \circ h_{(0)} \\
(60) &= \sum (X^1 Y_1^1 h_1 g^1 S(q^2 Y_2^2) Y^3 \triangleright h') \circ X^2 Y_2^1 h_2 g^2 S(X^3 q^1 Y_1^2) \\
(59, 58) &= \sum Z^1 X_1^1 Y_{(1,1)}^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_1 Y_1^3 h' \\
&\quad S(x^1 Z^2 X_2^1 Y_{(1,2)}^1 h_{(1,2)} g_2^1 S(q^2 Y_2^2)_2 Y_2^3) \\
&\quad \alpha x^2 Z_1^3 X^2 Y_2^1 h_2 g^2 S(x^3 Z_2^3 X^3 q^1 Y_1^2) \\
(3, 5) &= \sum Z^1 Y_{(1,1)}^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_1 Y_1^3 h' \\
&\quad S(Z^2 Y_{(1,2)}^1 h_{(1,2)} g_2^1 S(q^2 Y_2^2)_2 Y_2^3) \\
&\quad \alpha Z^3 Y_2^1 h_2 g^2 S(q^1 Y_1^2) \\
(11) &= \sum Z^1 [Y^1 h S(Y^2)]_{(1,1)} g_1^1 S(q^2)_1 Y_1^3 h' \\
&\quad S(Z^2 [Y^1 h S(Y^2)]_{(1,2)} g_2^1 S(q^2)_2 Y_2^3) \\
&\quad \alpha Z^3 [Y^1 h S(Y^2)]_2 g^2 S(q^1) \\
(1, 5) &= \sum Y^1 h S(Y^2) Z^1 g_1^1 S(q^2)_1 Y_1^3 h' S(Z^2 g_2^1 S(q^2)_2 Y_2^3) \alpha Z^3 g^2 S(q^1) \\
(62) &= \sum Y^1 h S(Y^2) g^1 S(X^3) f^1 S(q^2)_1 Y_1^3 h' \\
&\quad S(g_1^2 G^1 S(X^2) f^2 S(q^2)_2 Y_2^3) \alpha g_2^2 G^2 S(q^1 X^1) \\
(5, 63) &= \sum Y^1 h S(X^3 Y^2) f^1 S(q^2)_1 Y_1^3 h' S(q^1 X^1 \beta S(X^2) f^2 S(q^2)_2 Y_2^3) \\
(11) &= \sum Y^1 h S(q_2^2 X^3 Y^2) f^1 Y_1^3 h' S(q^1 X^1 \beta S(q_1^2 X^2) f^2 Y_2^3) \\
(64) &= \sum Y^1 h S(x^1 Y^2) \alpha x^2 Y_1^3 h' S(x^3 Y_2^3) \\
(58) &= h \circ h'.
\end{aligned}$$

□

5. HOPF MODULES IN ${}^H_H\mathcal{YD}$. INTEGRALS

Let H be a quasi-Hopf algebra. The aim of this Section is to define the space of integrals of a finite dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$, and to prove, following [24], [12], that it is an object of ${}^H_H\mathcal{YD}$, and that it has dimension 1. We will apply our results to the braided Hopf algebra associated to H , in the case where H is a quasitriangular quasi-Hopf algebra.

Let A be an algebra in a monoidal category \mathcal{C} . Recall that a right A -module M is an object $M \in \mathcal{C}$ together with a morphism $\underline{\omega}_M : M \otimes A \rightarrow M$ in \mathcal{C} such that $\underline{\omega}_M \circ (id_M \otimes \underline{\eta}) = l_M^{-1}$ and the following diagram is commutative:

$$\begin{array}{ccccc}
(M \otimes A) \otimes A & \xrightarrow{\underline{\omega}_M \otimes id_A} & M \otimes A & \xrightarrow{\underline{\omega}_M} & M \\
a_{M,A,A} \downarrow & & & & \uparrow \underline{\omega}_M \\
M \otimes (A \otimes A) & \xrightarrow{id_M \otimes \underline{m}} & M \otimes A & &
\end{array}$$

Clearly A itself is a right A -module, by right multiplication. Right comodules over a coalgebra C in \mathcal{C} can be defined in a similar way: we need $N \in \mathcal{C}$ together with a morphism $\underline{\rho}_N : N \rightarrow N \otimes C$ in \mathcal{C} such that $(id_N \otimes \underline{\varepsilon}) \circ \underline{\rho}_N = l_N$ and the following diagram is commutative:

$$\begin{array}{ccccc}
 N & \xrightarrow{\underline{\rho}_N} & N \otimes C & \xrightarrow{\underline{\rho}_N \otimes id_C} & (N \otimes C) \otimes C \\
 \downarrow \underline{\rho}_N & & & & \downarrow a_{N,C,C} \\
 N \otimes C & \xrightarrow{id_N \otimes \underline{\Delta}} & N \otimes (C \otimes C) & &
 \end{array}$$

C itself is a right C -comodule via the comultiplication $\underline{\Delta}$.

From [3], [21], [24], we recall the following.

Definition 5.1. Let B be a bialgebra in a braided category \mathcal{C} . A right B -Hopf module is a triple $(M, \underline{\omega}_M, \underline{\rho}_M)$, where $(M, \underline{\omega}_M)$ is a right B -module and $(M, \underline{\rho}_M)$ is a right B -comodule such that $\underline{\rho}_M : M \rightarrow M \otimes B$ is right B -linear. The B -module structure $\underline{\omega}_{M \otimes B} : (M \otimes B) \otimes B \rightarrow M \otimes B$ on $M \otimes B$ is given by the following composition:

$$\begin{aligned}
 (M \otimes B) \otimes B & \xrightarrow{id_{M \otimes B} \otimes \underline{\Delta}} (M \otimes B) \otimes (B \otimes B) \\
 & \xrightarrow{a_{M,B,B \otimes B}} M \otimes (B \otimes (B \otimes B)) \\
 & \xrightarrow{id_M \otimes a_{B,B,B}^{-1}} M \otimes ((B \otimes B) \otimes B) \\
 & \xrightarrow{id_M \otimes (c_{B,B} \otimes id_B)} M \otimes ((B \otimes B) \otimes B) \\
 & \xrightarrow{id_M \otimes a_{B,B,B}} M \otimes (B \otimes (B \otimes B)) \\
 & \xrightarrow{a_{M,B,B \otimes B}^{-1}} (M \otimes B) \otimes (B \otimes B) \\
 & \xrightarrow{\underline{\omega}_M \otimes \underline{m}} M \otimes B
 \end{aligned} \tag{66}$$

\mathcal{M}_B^B will denote the category of right B -Hopf modules and morphisms in \mathcal{C} preserving the B -action and the corresponding B -coaction.

We can consider algebras, coalgebras, bialgebras and Hopf algebras in the braided category ${}^H_H\mathcal{YD}$ over a quasi-Hopf algebra H . More precisely, an algebra B in ${}^H_H\mathcal{YD}$ is an object $B \in {}^H_H\mathcal{YD}$ such that

- B is a left H -module algebra, i.e. B has a multiplication \underline{m} and a usual unit 1_B satisfying the following conditions:

$$(ab)c = \sum (X^1 \cdot a)[(X^2 \cdot b)(X^3 \cdot c)], \tag{67}$$

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B, \tag{68}$$

for all $a, b, c \in B$ and $h \in H$.

- B is a quasi-comodule algebra, that is, the multiplication \underline{m} and the unit $\underline{\eta}$ of B intertwine the H -coaction λ_B . By (47) this means:

$$\begin{aligned}
 \lambda_B(bb') &= \sum X^1(x^1 Y^1 \cdot b)_{(-1)} x^2(Y^2 \cdot b')_{(-1)} Y^3 \\
 &\quad \otimes [X^2 \cdot (x^1 Y^1 \cdot b)_{(0)}][X^3 x^3 \cdot (Y^2 \cdot b')_{(0)}],
 \end{aligned} \tag{69}$$

for all $b, b' \in B$, and

$$(70) \quad \lambda_B(1_B) = 1_H \otimes 1_B.$$

$M \in {}^H_H\mathcal{YD}$ is a right B -module if there exists a morphism $\underline{\omega}_M : M \otimes B \rightarrow M$ in ${}^H_H\mathcal{YD}$ (we will denote $\underline{\omega}_M(m \otimes b) := m \leftarrow b$) such that

$$(71) \quad m \leftarrow 1_B = m, \quad (m \leftarrow b) \leftarrow b' = \sum (X^1 \cdot m) \leftarrow [(X^2 \cdot b)(X^3 \cdot b')]$$

for all $m \in M, b, b' \in B$. The fact that $\underline{\omega}_M$ is a morphism in ${}^H_H\mathcal{YD}$ means (see (47))

$$(72) \quad h \cdot (m \leftarrow b) = \sum (h_1 \cdot m) \leftarrow (h_2 \cdot b),$$

$$(73) \quad \begin{aligned} \lambda_M(m \leftarrow b) &= \sum X^1(x^1 Y^1 \cdot m)_{(-1)} x^2(Y^2 \cdot b)_{(-1)} Y^3 \\ &\otimes [X^2 \cdot (x^1 Y^1 \cdot m)_{(0)}] \leftarrow [X^3 x^3 \cdot (Y^2 \cdot b)_{(0)}] \end{aligned}$$

for all $m \in M, b \in B$.

Similarly, $B \in {}^H_H\mathcal{YD}$ is a coalgebra if

- B is a left H -module coalgebra, i.e. B has a comultiplication $\underline{\Delta}_B : B \rightarrow B \otimes B$ (we will denote $\underline{\Delta}(b) = \sum b_{\underline{1}} \otimes b_{\underline{2}}$) and a usual counit $\underline{\varepsilon}_B$ such that:

$$(74) \quad \sum X^1 \cdot b_{(\underline{1}, \underline{1})} \otimes X^2 \cdot b_{(\underline{1}, \underline{2})} \otimes X^3 \cdot b_{\underline{2}} = \sum b_{\underline{1}} \otimes b_{(\underline{2}, \underline{1})} \otimes b_{(\underline{2}, \underline{2})},$$

$$(75) \quad \underline{\Delta}_B(h \cdot b) = \sum h_1 \cdot b_{\underline{1}} \otimes h_2 \cdot b_{\underline{2}}, \quad \underline{\varepsilon}_B(h \cdot b) = \varepsilon(h) \underline{\varepsilon}_B(b),$$

for all $h \in H, b \in B$, where we use the same notation for the quasi-coassociativity of $\underline{\Delta}_B$ as in Section 2.

- B is a quasi-comodule coalgebra, i.e. the comultiplication $\underline{\Delta}_B$ and the counit $\underline{\varepsilon}_B$ intertwine the H -coaction λ_B . Explicitly, for all $b \in B$ we must have that:

$$(76) \quad \begin{aligned} \sum b_{(-1)} \otimes b_{(0)\underline{1}} \otimes b_{(0)\underline{2}} &= \sum X^1(x^1 Y^1 \cdot b_{\underline{1}})_{(-1)} x^2(Y^2 \cdot b_{\underline{2}})_{(-1)} Y^3 \\ &\otimes X^2 \cdot (x^1 Y^1 \cdot b_{\underline{1}})_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot b_{\underline{2}})_{(0)}, \end{aligned}$$

and

$$(77) \quad \sum \varepsilon_B(b_{(0)}) b_{(-1)} = \varepsilon_B(b) 1.$$

A right B -comodule in ${}^H_H\mathcal{YD}$ is an object $M \in {}^H_H\mathcal{YD}$ together with a morphism $\underline{\rho}_M : M \rightarrow M \otimes B$ in ${}^H_H\mathcal{YD}$ (we will denote $\underline{\rho}_M(m) = \sum m_{(\underline{0})} \otimes m_{(\underline{1})}$ for all $m \in M$) such that the following relations hold, for all $m \in M$:

$$(78) \quad \sum X^1 \cdot m_{(\underline{0}, \underline{0})} \otimes X^2 \cdot m_{(\underline{0}, \underline{1})} \otimes X^3 \cdot m_{(\underline{1})} = \sum m_{(\underline{0})} \otimes m_{(\underline{1}, \underline{1})} \otimes m_{(\underline{1}, \underline{2})},$$

$$(79) \quad \sum \underline{\varepsilon}(m_{(\underline{1})}) m_{(\underline{0})} = m,$$

where we will denote

$$(\underline{\rho}_M \otimes id_B)(\underline{\rho}_M(m)) = \sum m_{(\underline{0}, \underline{0})} \otimes m_{(\underline{0}, \underline{1})} \otimes m_{(\underline{1})} \text{ etc.}$$

The fact that $\underline{\rho}_M$ is a morphism in ${}^H_H\mathcal{YD}$ means that (see (47))

$$(80) \quad \underline{\rho}_M(h \cdot m) = \sum h_1 \cdot m_{(\underline{0})} \otimes h_2 \cdot m_{(\underline{1})},$$

and

$$\begin{aligned} \sum m_{(-1)} \otimes m_{(0)(\underline{0})} \otimes m_{(0)(\underline{1})} &= \sum X^1(x^1 Y^1 \cdot m_{(\underline{0})})_{(-1)} x^2 (Y^2 \cdot m_{(\underline{1})})_{(-1)} Y^3 \\ (81) \quad &\otimes X^2 \cdot (x^1 Y^1 \cdot m_{(\underline{0})})_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot m_{(\underline{1})})_{(0)}, \end{aligned}$$

for all $h \in H$ and $m \in M$.

Now, a bialgebra $B \in {}^H_H \mathcal{YD}$ is an algebra and a coalgebra in ${}^H_H \mathcal{YD}$ such that $\underline{\Delta}_B$ is an algebra morphism, i.e. $\underline{\Delta}_B(1_B) = 1_B \otimes 1_B$ and, by (38) and (48), for all $b, b' \in B$ we have that:

$$\begin{aligned} \Delta_B(bb') &= \sum [y^1 X^1 \cdot b_{\underline{1}}] [y^2 Y^1 (x^1 X^2 \cdot b_{\underline{2}})_{(-1)} x^2 X_1^3 \cdot b'_{\underline{1}}] \\ (82) \quad &\otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot b_{\underline{2}})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b'_{\underline{2}}]. \end{aligned}$$

If $B \in {}^H_H \mathcal{YD}$ is a bialgebra then $M \in {}^H_H \mathcal{YD}$ is a right B -Hopf module if M is a right B -module (as above, we will denote $\varpi_M(m \otimes b) = m \leftarrow b$) and a right B -comodule such that the right B -coaction on M , $\rho_M : M \rightarrow M \otimes B$, is right B -linear, which means that the following relation holds, for all $m \in M$ and $b \in B$ (see (66)):

$$\begin{aligned} \rho_M(m \leftarrow b) &= \sum (y^1 X^1 \cdot m_{(\underline{0})}) \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m_{(\underline{1})})_{(-1)} x^2 X_1^3 \cdot b_{\underline{1}}] \\ (83) \quad &\otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot m_{(\underline{1})})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b_{\underline{2}}]. \end{aligned}$$

Finally, a bialgebra B in ${}^H_H \mathcal{YD}$ is a braided Hopf algebra if there exists a morphism $\underline{S} : B \rightarrow B$ in ${}^H_H \mathcal{YD}$ such that $\sum \underline{S}(b_{\underline{1}}) b_{\underline{2}} = \sum b_{\underline{1}} \underline{S}(b_{\underline{2}}) = \varepsilon(b) 1_B$, for all $b \in B$. Since \underline{S} is a morphism in ${}^H_H \mathcal{YD}$, we have that

$$(84) \quad \underline{S}(h \cdot b) = h \cdot \underline{S}(b) \quad \text{and} \quad \sum \underline{S}(b)_{(-1)} \otimes \underline{S}(b)_{(0)} = \sum b_{(-1)} \otimes \underline{S}(b_{(0)}),$$

for all $h \in H$, $b \in B$. Also, by (39) and (48) we obtain that

$$(85) \quad \underline{S}(bb') = \sum [b_{(-1)} \cdot \underline{S}(b')] \underline{S}(b_{(0)}) \quad \text{and} \quad \underline{\Delta}(\underline{S}(b)) = \sum b_{\underline{1}(-1)} \cdot \underline{S}(b_{\underline{2}}) \otimes \underline{S}(b_{\underline{1}(0)}),$$

for all $b, b' \in B$.

The first step to prove the existence and uniqueness of integrals in a finite dimensional braided Hopf algebra is the structure theorem for Hopf modules. To this end we need first the following result.

Lemma 5.2. *Let H be a quasi-bialgebra, B a bialgebra in ${}^H_H \mathcal{YD}$ and $N \in {}^H_H \mathcal{YD}$. Then $N \otimes B \in \mathcal{M}_B^B$ with following action $\varpi_{N \otimes B} : (N \otimes B) \otimes B \rightarrow N \otimes B$ and coaction $\rho_{N \otimes B} : N \otimes B \rightarrow (N \otimes B) \otimes B$ given by*

$$(86) \quad (n \otimes b) \prec b' = \sum X^1 \cdot n \otimes [(X^2 \cdot b)(X^3 \cdot b')],$$

$$(87) \quad \rho_{N \otimes B}(n \otimes b) := \sum x^1 \cdot n \otimes x^2 \cdot b_{\underline{1}} \otimes x^3 \cdot b_{\underline{2}},$$

for all $n \in N$ and $b, b' \in B$.

Proof. ${}^H_H \mathcal{YD}$ is a braided category, so $N \otimes B \in {}^H_H \mathcal{YD}$. It is not hard to see that (1) and (67) imply that $\varpi_{N \otimes B}$ is left H -linear. It intertwines also the corresponding H -coaction. Indeed, by (47), the left H -coaction on $(N \otimes B) \otimes B$ is given by

$$\begin{aligned} &\lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b') \\ &= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \\ &\quad \otimes Z_1^2 X^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \otimes Z_2^2 X^3 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)} \otimes Z^3 y^3 \cdot (T^2 \cdot b')_{(0)}, \end{aligned}$$

for all $n \in N$, $b, b' \in B$. Therefore:

$$\begin{aligned}
& (id_H \otimes \underline{\omega}_{N \otimes B}) \circ \lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b') \\
(86) \quad &= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} \\
& \quad Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \otimes W^1 Z_1^2 X^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes [(W^2 Z_2^2 X^3 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)})] [(W^3 Z^3 y^3 \cdot (T^2 \cdot b')_{(0)})] \\
(3, 45, 67) \quad &= \sum Z^1 (X_1^1 x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} X_2^1 x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} \\
& \quad Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \otimes Z^2 \cdot (X_1^1 x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes Z^3 \cdot [(X^2 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)}) (X^3 y^3 \cdot (T^2 \cdot b')_{(0)})] \\
(3) \text{ twice, (45)} \quad &= \sum Z^1 (x^1 Y^1 T_1^1 \cdot n)_{(-1)} x^2 X^1 (y^1 Y^2 T_2^1 \cdot b)_{(-1)} y^2 \\
& \quad (Y_1^3 T^2 \cdot b')_{(-1)} Y_2^3 T^3 \otimes Z^2 \cdot (x^1 Y^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes Z^3 x^3 \cdot [(X^2 \cdot (y^1 Y^2 T_2^1 \cdot b)_{(0)}) (X^3 y^3 \cdot (Y_1^3 T^2 \cdot b')_{(0)})] \\
(3, 69) \quad &= \sum Z^1 (x^1 Y^1 T^1 \cdot n)_{(-1)} x^2 [(Y_1^2 T^2 \cdot b) (Y_2^2 T^3 \cdot b')_{(-1)} Y^3 \\
& \quad \otimes Z^2 \cdot (x^1 Y^1 T^1 \cdot n)_{(0)} \otimes Z^3 x^3 \cdot [(Y_1^2 T^2 \cdot b) (Y_2^2 T^3 \cdot b')_{(0)}] \\
(47, 86) \quad &= \sum \lambda_{N \otimes B} (T^1 \cdot n \otimes (T^2 \cdot b) (T^3 \cdot b')) \\
&= \lambda_{N \otimes B} \circ \underline{\omega}_{N \otimes B}((n \otimes b) \otimes b')
\end{aligned}$$

for all $n \in N$ and $b, b' \in B$. In a similar way, it can be proved that the map $\underline{\rho}_{N \otimes B}$ is a morphism in ${}^H_H \mathcal{YD}$, we leave it to the reader to verify the details.

Using (67) and (3), it easily follows that $N \otimes B$ is a right B -module. Also, it is not hard to see that (74), (75) and (3) imply that $N \otimes B$ is a right B -comodule. It remains only to show that $\underline{\rho}_{N \otimes B}$ is right B -linear. By (66), we have that the right B -module structure of $(N \otimes B) \otimes B$ is given by

$$\begin{aligned}
& [(n \otimes b) \otimes b'] \bullet b'' \\
&= \sum [Z^1 y_1^1 X_1^1 \cdot n \otimes (Z^2 y_2^1 X_2^1 \cdot b) (Z^3 y^2 Y^1 (x^1 X^2 \cdot b')_{(-1)} x^2 X_1^3 \cdot b''_{\underline{1}})] \\
& \quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot b')_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b''_{\underline{2}}],
\end{aligned}$$

for all $n \in N$ and $b, b', b'' \in B$. This allows us to compute, for any $n \in N$ and $b, b' \in B$, that:

$$\begin{aligned}
\underline{\rho}_{N \otimes B} (n \otimes b) \bullet b' &= \sum [(z^1 \cdot n \otimes z^2 \cdot b_{\underline{1}}) \otimes z^3 \cdot b_{\underline{2}}] \bullet b' \\
&= \sum [Z^1 y_1^1 X_1^1 z^1 \cdot n \\
& \quad \otimes (Z^2 y_2^1 X_2^1 z^2 \cdot b_{\underline{1}}) (Z^3 y^2 Y^1 (x^1 X^2 z^3 \cdot b_{\underline{2}})_{(-1)} x^2 X_1^3 \cdot b'_{\underline{1}})] \\
& \quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 z^3 \cdot b_{\underline{2}})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b'_{\underline{2}}]
\end{aligned}$$

$$\begin{aligned}
(3) &= \sum [Z^1 y_1^1 z^1 X^1 \cdot n \otimes (Z^2 y_2^1 z^2 T^1 X_1^2 \cdot b_{\underline{1}}) \\
&\quad (Z^3 y^2 Y^1 (x^1 z_1^3 T^2 X_2^2 \cdot b_{\underline{2}})_{(-1)} x^2 z_{(2,1)}^3 T_1^3 X_1^3 \cdot b'_{\underline{1}})] \\
&\quad \otimes [y_1^3 Y^2 \cdot (x^1 z_1^3 T^2 X_2^2 \cdot b_{\underline{2}})_{(0)}] [y_2^3 Y^3 x^3 z_{(2,2)}^3 T_2^3 X_2^3 \cdot b'_{\underline{2}}] \\
(1, 45) &= \sum [Z^1 y_1^1 z^1 X^1 \cdot n \otimes (Z^2 y_2^1 z^2 T^1 X_1^2 \cdot b_{\underline{1}}) \\
&\quad (Z^3 y^2 Y^1 z_{(1,1)}^3 (x^1 T^2 X_2^2 \cdot b_{\underline{2}})_{(-1)} x^2 T_1^3 X_1^3 \cdot b'_{\underline{1}})] \\
&\quad \otimes [y_1^3 Y^2 z_{(1,2)}^3 \cdot (x^1 T^2 X_2^2 \cdot b_{\underline{2}})_{(0)}] [y_2^3 Y^3 z_2^3 x^3 T_2^3 X_2^3 \cdot b'_{\underline{2}}] \\
(1, 3, 67) &= \sum \{y^1 X^1 \cdot n \otimes y^2 \cdot [(z^1 T^1 X_1^1 \cdot b_{\underline{1}}) \\
&\quad (z^2 Y^1 (x^1 T^2 X_2^2 \cdot b_{\underline{2}})_{(-1)} x^2 T_1^3 X_1^3 \cdot b'_{\underline{1}})]\} \\
&\quad \otimes y^3 \cdot \{[z_1^3 Y^2 \cdot (x^1 T^2 X_2^2 \cdot b_{\underline{2}})_{(0)}] [z_2^3 Y^3 x^3 T_2^3 X_2^3 \cdot b'_{\underline{2}}]\} \\
(75, 82) &= \sum \{y^1 X^1 \cdot n \otimes y^2 \cdot [(X^2 \cdot b)(X^3 \cdot b')]_{\underline{1}}\} \otimes y^3 \cdot [(X^2 \cdot b)(X^3 \cdot b')]_{\underline{2}} \\
(87, 86) &= \sum \rho_{N \otimes B}(X^1 \cdot n \otimes (X^2 \cdot b)(X^3 \cdot b')) = \rho_{N \otimes B}((n \otimes b) \prec b'),
\end{aligned}$$

as needed. \square

Our next result is the Fundamental Theorem for Hopf modules in the braided monoidal category ${}^H_H \mathcal{YD}$, generalizing [12, Theorem 1].

Theorem 5.3. *Let H be a quasi-Hopf algebra, B a Hopf algebra in ${}^H_H \mathcal{YD}$ and $M \in \mathcal{M}_B^B$.*

- (i) $M^{\text{co}B} = \{m \in M \mid \rho_M(m) = m \otimes 1_B\} \in {}^H_H \mathcal{YD}$.
- (ii) For all $m \in M$, we have that $P(m) = \sum m_{(\underline{0})} \leftarrow \underline{S}(m_{(\underline{1})}) \in M^{\text{co}B}$.
- (iii) $\rho_M(n \leftarrow b) = \sum (x^1 \cdot n) \leftarrow (x^2 \cdot b_{\underline{1}}) \otimes x^3 \cdot b_{\underline{2}}$ and $P(n \leftarrow b) = \underline{\varepsilon}(b)n$, for all $n \in M^{\text{co}B}$ and $b \in B$.
- (iv) The map

$$F: M^{\text{co}B} \otimes B \rightarrow M, \quad F(n \otimes b) = n \leftarrow b,$$

is an isomorphism of Hopf modules in ${}^H_H \mathcal{YD}$, with inverse G given by

$$G(m) = \sum P(m_{(\underline{0})}) \otimes m_{(\underline{1})}.$$

Proof. (i) If $n \in M^{\text{co}B}$, then $\rho_M(h \cdot n) = \sum h_1 \cdot n \otimes h_2 \cdot 1_B = h \cdot n \otimes 1_B$, by (72) and (67). This shows that $M^{\text{co}B}$ is an H -submodule of M . On the other hand, for any $n \in N$ we have

$$\begin{aligned}
&\sum n_{(-1)} \otimes n_{(0)(\underline{0})} \otimes n_{(0)(\underline{1})} \\
(81) &= \sum X^1 (x^1 Y^1 \cdot n)_{(-1)} x^2 (Y^2 \cdot 1_B)_{(-1)} Y^3 \\
&\quad \otimes X^2 \cdot (x^1 Y^1 \cdot n)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot 1_B)_{(0)} \\
(67) \text{ twice, } (70) &= \sum n_{(-1)} \otimes n_{(0)} \otimes 1_B.
\end{aligned}$$

Thus, $\rho_M(n) = \sum n_{(-1)} \otimes n_{(0)} \in H \otimes M^{\text{co}B}$ which means that $M^{\text{co}B}$ is a left H -quasi-subcomodule of M . It follows from the above arguments that $M^{\text{co}B} \in {}^H_H \mathcal{YD}$.

(ii) For any $m \in M$, we have that

$$\begin{aligned}
\rho_M(P(m)) &= \sum \rho_M(m_{(\underline{0})} \leftarrow \underline{S}(m_{(\underline{1})})) \\
(83) \quad &= \sum (y^1 X^1 \cdot m_{(\underline{0}, \underline{0})} \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m_{(\underline{0}, \underline{1})})_{(-1)} x^2 X_1^3 \cdot \underline{S}(m_{(\underline{1})})_{\underline{1}}] \\
&\quad [y_1^3 Y^2 \cdot (x^1 X^2 \cdot m_{(\underline{0}, \underline{1})})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot \underline{S}(m_{(\underline{1})})_{\underline{2}}]) \\
(78, 84, 85) &= \sum (y^1 \cdot m_{(\underline{0})} \leftarrow \\
&\quad [y^2 Y^1 (x^1 \cdot m_{(\underline{1}, \underline{1})})_{(-1)} x^2 \underline{S}(m_{(\underline{1}, \underline{2}, \underline{1})})_{(-1)} \cdot \underline{S}(m_{(\underline{1}, \underline{2}, \underline{2})})] \\
&\quad \otimes y^3 \cdot \{[Y^2 \cdot (x^1 \cdot m_{(\underline{1}, \underline{1})})_{(0)}] [Y^3 x^3 \cdot \underline{S}(m_{(\underline{1}, \underline{2}, \underline{1})})_{(0)}]\}) \\
(69) &= \sum (y^1 \cdot m_{(\underline{0})} \leftarrow \\
&\quad y^2 [(x^1 \cdot m_{(\underline{1}, \underline{1})}) (x^2 \cdot \underline{S}(m_{(\underline{1}, \underline{2}, \underline{1})}))]_{(-1)} x^3 \cdot \underline{S}(m_{(\underline{1}, \underline{2}, \underline{2})}) \\
&\quad \otimes y^3 \cdot [(x^1 \cdot m_{(\underline{1}, \underline{1})}) (x^2 \cdot \underline{S}(m_{(\underline{1}, \underline{2}, \underline{1})}))]_{(0)}) \\
(84, 74, 67) &= \sum (y^1 \cdot m_{(\underline{0})} \leftarrow (y^2 \cdot \underline{S}(m_{(\underline{1})})) \otimes y^3 \cdot 1_B = P(m) \otimes 1_B.
\end{aligned}$$

(iii) For all $n \in N$ and $b \in B$, we compute, using (83),

$$\begin{aligned}
\rho_M(n \leftarrow b) &= \sum (y^1 X^1 \cdot n \leftarrow [y^2 Y^1 (x^1 X^2 \cdot 1_B)_{(-1)} x^2 X_1^3 \cdot b_{\underline{1}}] \\
&\quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot 1_B)_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b_{\underline{2}}]) \\
(67, 70) &= \sum (y^1 \cdot n \leftarrow (y^2 \cdot b_{\underline{1}}) \otimes y^3 \cdot b_{\underline{2}}).
\end{aligned}$$

For all $n \in M^{\text{co}B}$, we find

$$\begin{aligned}
P(n \leftarrow b) &= \sum [(y^1 \cdot n) \leftarrow (y^2 \cdot b_{\underline{1}})] \leftarrow \underline{S}(y^3 \cdot b_{\underline{2}}) \\
(71, 84) &= \sum n \leftarrow b_{\underline{1}} \underline{S}(b_{\underline{2}}) = \underline{\varepsilon}(b) n \leftarrow 1_B = \underline{\varepsilon}(b) n.
\end{aligned}$$

(iv) By (i) and Lemma 5.2, we obtain that $M^{\text{co}B} \otimes B \in \mathcal{M}_B^B$. It follows from (72) that F is left H -linear. It also intertwines the corresponding left H -coaction by (47) and (73). Now we will prove that F and G are inverses. For all $m \in M$, we have

$$\begin{aligned}
FG(m) &= \sum P(m_{(\underline{0})}) \leftarrow m_{(\underline{1})} \\
(71) &= \sum (X^1 \cdot m_{(\underline{0}, \underline{0})} \leftarrow [(X^2 \cdot \underline{S}(m_{(\underline{0}, \underline{1})})) (X^3 \cdot m_{(\underline{1})})] \\
(84, 78, 79) &= \sum m_{(\underline{0})} \leftarrow \underline{S}(m_{(\underline{1}, \underline{1})}) m_{(\underline{1}, \underline{2})} = m \leftarrow 1_B = m.
\end{aligned}$$

Similarly, for any $n \in M^{\text{co}B}$ and $b \in B$, we compute

$$\begin{aligned}
GF(n \otimes b) &= \sum P((n \leftarrow b)_{(\underline{0})}) \otimes (n \leftarrow b)_{(\underline{1})} \\
(iii) &= \sum P((x^1 \cdot n) \leftarrow (x^2 \cdot b_{\underline{1}})) \otimes x^3 \cdot b_{\underline{2}} \\
(iii), (75) &= \sum P(n) \otimes b = n \otimes b.
\end{aligned}$$

We are left to show that F is a morphism in \mathcal{M}_B^B . It is not hard to see that (86) and (71) imply that F is right B -linear. Also, (iii) implies that

$$\rho_M \circ F(n \otimes b) = (F \otimes id_B) \circ \rho_{M^{\text{co}B} \otimes B}(n \otimes b) = \sum (x^1 \cdot n) \leftarrow (x^2 \cdot b_{\underline{1}}) \otimes x^3 \cdot b_{\underline{2}},$$

for all $n \in N$ and $b \in B$, and this finishes the proof. \square

Let H be a quasi-Hopf algebra, and let ${}^H_H\mathcal{YD}^{\text{fd}}$ be the category of finite dimensional left Yetter-Drinfeld modules over H . If $M \in {}^H_H\mathcal{YD}^{\text{fd}}$, then $M^* \in {}^H_H\mathcal{YD}^{\text{fd}}$ (cf. [4]). The action and coaction are given by

$$(88) \quad (h \cdot m^*)(m) = m^*(S(h) \cdot m)$$

$$(89) \quad \lambda_{M^*}(m^*) = \sum m_{(-1)}^* \otimes m_{(0)}^* = \sum_{i=1}^n \langle m^*, f^2 \cdot (g^1 \cdot {}^i m)_{(0)} \rangle$$

$$S^{-1}(f^1(g^1 \cdot {}^i m)_{(-1)}g^2) \otimes {}^i m$$

for all $h \in H$, $m^* \in M^*$, $m \in M$. Here $f = \sum f^1 \otimes f^2$ is the twist defined in (15), $({}^i m)_{i=\overline{1,n}}$ is a basis of M and $({}^i m)_{i=\overline{1,n}}$ its dual basis. Moreover, ${}^H_H\mathcal{YD}^{f.d.}$ is a rigid monoidal category. For each object $M \in {}^H_H\mathcal{YD}^{\text{fd}}$, the evaluation and coevaluation maps (ev_M and $coev_M$, respectively) are given by (41).

In addition, if $B \in {}^H_H\mathcal{YD}^{\text{fd}}$ is a Hopf algebra, then B^* is a Hopf algebra in ${}^H_H\mathcal{YD}^{f.d.}$. The structure is the following.

- The multiplication and unit are given by

$$(90) \quad (\varphi * \psi)(b) = \langle \varphi, f^2 \tilde{q}_2^2 Y^3 S^{-1}(\tilde{q}^1 Y^1 (p^1 \cdot b_{\underline{2}})_{(-1)} p^2) \cdot b_{\underline{1}} \rangle$$

$$\langle \psi, f^1 \tilde{q}_1^2 Y^2 \cdot (p^1 \cdot b_{\underline{2}})_{(0)} \rangle,$$

$$(91) \quad 1_{B^*} = \underline{\varepsilon}$$

for all $\varphi, \psi \in B^*$, $b \in B$, where $q_L = \sum \tilde{q}^1 \otimes \tilde{q}^2$ and $p_R = \sum p^1 \otimes p^2$ are the elements defined in (21) and (19).

- the comultiplication and counit are given by the formulas

$$(92) \quad \underline{\Delta}_{B^*}(\varphi) = \sum_{i,j=1}^n \langle \varphi, [(g^1 \cdot {}^j b)_{(-1)} g^2 \cdot {}^i b] (g^1 \cdot {}^j b)_{(0)} \rangle {}^i b \otimes {}^j b$$

$$(93) \quad \underline{\varepsilon}_{B^*}(\varphi) = \varphi(1_B),$$

for any $\varphi \in B^*$, where $f^{-1} = \sum g^1 \otimes g^2$ was defined in (16), $({}^i b)_{i=\overline{1,n}}$ is a basis of B and $({}^i b)_{i=\overline{1,n}}$ the corresponding dual basis of B^* .

- the antipode is given by

$$(94) \quad \underline{S}_{B^*} = \underline{S}^*, \text{ i. e. } \underline{S}_{B^*}(\varphi) = \varphi \circ \underline{S},$$

for all $\varphi \in B^*$.

Proposition 5.4. *Let $B \in {}^H_H\mathcal{YD}^{\text{fd}}$ a Hopf algebra. Then B^* is a right B -Hopf module, with structure:*

$$(95) \quad \langle \varphi \leftarrow b, b' \rangle = \sum \langle \varphi, [(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}((U^1 \cdot b)_{(0)}) \rangle,$$

$$(96) \quad \underline{\rho}_{B^*}(\varphi) = \sum_{i=1}^n (S(\tilde{p}^1) \cdot {}^i b)_{(-1)} \cdot [{}^i b * (\tilde{p}^2 \cdot \varphi)] \otimes (S(\tilde{p}^1) \cdot {}^i b)_{(0)},$$

for all $\varphi \in B^*$, $b, b' \in B$, where

$$(97) \quad U = \sum U^1 \otimes U^2 := \sum g^1 S(q^2) \otimes g^2 S(q^1),$$

$p_L = \sum \tilde{p}^1 \otimes \tilde{p}^2$, $q_R = \sum q^1 \otimes q^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are the elements defined by (21), (19) and (16), and $\{i b\}_{i=\overline{1,n}}$ is a basis of B with corresponding dual basis $\{i b\}_{i=\overline{1,n}}$. Moreover,

$$B^{*\text{co}B} = \{\Lambda \in B^* \mid \sum (\tilde{p}^1 \cdot \varphi) * (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B)\Lambda \text{ for all } \varphi \in B^*\}.$$

Proof. If B is a Hopf algebra in a braided rigid monoidal category \mathcal{C} , then B^* is a right Hopf B -module, as follows.

- the right B -module structure $\lrcorner: B^* \otimes B \rightarrow B$ on B^* is the composition

$$(98) \quad \begin{array}{ccc} B^* \otimes B & \xrightarrow{l_{B^* \otimes B}} & (B^* \otimes B) \otimes \underline{1} \\ & \xrightarrow{(id_{B^*} \otimes \underline{S}) \otimes coev_B} & (B^* \otimes B) \otimes (B \otimes B^*) \\ & \xrightarrow{a_{B^* \otimes B, B, B^*}^{-1}} & ((B^* \otimes B) \otimes B) \otimes B^* \\ & \xrightarrow{a_{B^*, B, B} \otimes id_{B^*}} & (B^* \otimes (B \otimes B)) \otimes B^* \\ & \xrightarrow{(id_{B^*} \otimes \underline{m}) \otimes id_{B^*}} & (B^* \otimes B) \otimes B^* \\ & \xrightarrow{ev_B \otimes id_{B^*}} & \underline{1} \otimes B^* \\ & \xrightarrow{r_{B^*}^{-1}} & B^* \end{array}$$

- the right B -comodule structure $\rho_{B^*}: B^* \rightarrow B^* \otimes B$ on B^* is the composition

$$(99) \quad \begin{array}{ccccc} B^* & \xrightarrow{r_{B^*}} & \underline{1} \otimes B^* & \xrightarrow{coev_B \otimes id_{B^*}} & (B \otimes B^*) \otimes B^* \\ & \xrightarrow{a_{B, B^*, B^*}} & B \otimes (B^* \otimes B^*) & \xrightarrow{id_B \otimes \underline{m}_{B^*}} & B \otimes B^* \\ & \xrightarrow{c_{B, B^*}} & B^* \otimes B. & & \end{array}$$

Let $\gamma = \sum \gamma^1 \otimes \gamma^2$ and $f^{-1} = \sum g^1 \otimes g^2$ be the elements defined in (14) and (16). By (98), we have, for all $\varphi \in B^*$ and $b, b' \in B$:

$$\begin{aligned} & \langle \varphi \lrcorner b, b' \rangle \\ &= \sum \langle \varphi, S(X^1 p_1^1) \alpha \cdot [((X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} X^3 p^2 \cdot b') (X^2 p_2^1 \cdot \underline{S}(b))_{(0)}] \rangle \\ (67, 45) \quad &= \sum \langle \varphi, [(S(X^1 p_1^1)_1 \alpha_1 X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} S(X^1 p_1^1)_2 \alpha_2 X^3 p^2 \cdot b'] \\ & \quad (S(X^1 p_1^1)_1 \alpha_1 X^2 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \\ (17, 11) \quad &= \sum \langle \varphi, [(g^1 S(X_2^1 p_{(1,2)}^1) \gamma^1 X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} g^2 S(X_1^1 p_{(1,1)}^1) \gamma^2 X^3 p^2 \cdot b'] \\ & \quad (g^1 S(X_2^1 p_{(1,2)}^1) \gamma^1 X^2 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \\ (14, 3, 5) \quad &= \sum \langle \varphi, g^1 S(Y^2 p_{(1,2)}^1) \alpha Y^3 p_2^1 \cdot \underline{S}(b))_{(-1)} g^2 S(Y^1 p_{(1,1)}^1) \alpha p^2 \cdot b'] \\ & \quad (g^1 S(Y^2 p_{(1,2)}^1) \alpha Y^3 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \\ (1, 5, 6) \quad &= \sum \langle \varphi, [(g^1 S(Y^2) \alpha Y^3 \cdot \underline{S}(b))_{(-1)} g^2 S(Y^1) \cdot b'] \\ & \quad (g^1 S(Y^2) \alpha Y^3 \cdot \underline{S}(b))_{(0)} \rangle \\ (19, 97, 84) \quad &= \sum \langle \varphi, [\underline{S}(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}(U^1 \cdot b)_{(0)} \rangle \\ (84) \quad &= \sum \langle \varphi, [(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}((U^1 \cdot b)_{(0)}) \rangle \end{aligned}$$

which is just (95). (96) follows easily by (99), the details are left to the reader. Finally, by (99) we have

$$\begin{aligned}
\Lambda \in B^{*co(B)} &\iff \rho_{B^*}(\Lambda) = \Lambda \otimes 1_B \\
&\iff c_{B,B^*}^{-1} \circ \rho_{B^*}(\Lambda) = c_{B,B^*}^{-1}(\Lambda \otimes 1_B) \\
&\iff \sum_{i=1}^n S(\tilde{p}^1) \cdot {}_i b \otimes {}^i b * (\tilde{p}^2 \cdot \Lambda) = 1_B \otimes \Lambda \\
&\iff \sum (\tilde{p}^1 \cdot \varphi) * (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B)\Lambda, \text{ for all } \varphi \in B^*.
\end{aligned}$$

□

We define the space of left integrals by $I_l(B^*) = B^{*co(B)}$. From the Fundamental Theorem for Hopf modules, we then obtain.

Corollary 5.5. *Let H be a quasi-Hopf algebra and B a finite dimensional Hopf algebra in ${}^H_H\mathcal{YD}$. Then $I_l(B^*) \otimes B \simeq B^*$ as right B -Hopf modules. In particular, $\dim_k(I_l(B^*)) = 1$.*

Now, let H be a quasi-Hopf algebra and H_0 the H -module algebra described in Section 4. If (H, R) is quasitriangular, then H_0 is a Hopf algebra in ${}^H_H\mathcal{YD}$, see [5]. The additional structure is the following.

$$(100) \quad \lambda_{H_0}(h) = \sum R^2 \otimes R^1 \triangleright h,$$

$$(101) \quad \underline{\Delta}(h) = \sum h_{\underline{1}} \otimes h_{\underline{2}}$$

$$(102) \quad = \sum x^1 X^1 h_1 g^1 S(x^2 R^2 y^3 X_2^3) \otimes x^3 R^1 \triangleright y^1 X^2 h_2 g^2 S(y^2 X_1^3),$$

$$(103) \quad \underline{\varepsilon}(h) = \varepsilon(h),$$

$$(104) \quad \underline{S}(h) = \sum X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright h) S(q^2) X^3),$$

for all $h \in H$, where $R = \sum R^1 \otimes R^2$ and $f^{-1} = \sum g^1 \otimes g^2$, $p_R = \sum p^1 \otimes p^2$ and $q_R = \sum q^1 \otimes q^2$ are the elements defined by (16), (19) and (20). By the above arguments, if H is a finite dimensional Hopf algebra, then H_0^* is also a Hopf algebra in ${}^H_H\mathcal{YD}$, with structure

$$\begin{aligned}
(105) \quad (\varphi * \Psi)(h) &= \sum \langle \varphi, f^2 \overline{R}^2 \triangleright h_{\underline{1}} \rangle \langle \Psi, f^1 \overline{R}^1 \triangleright h_{\underline{2}} \rangle \\
&= \sum \langle \varphi, f^2 \triangleright Y^2 \overline{R}^2 X^1 x_1^1 h_1 g^1 S(Y^3 x^3) \rangle
\end{aligned}$$

$$(106) \quad \langle \Psi, f^1 Y^1 \overline{R}^1 \triangleright X^2 x_2^1 h_2 g^2 S(X^3 x^2) \rangle,$$

$$(107) \quad 1_{H_0^*} = \underline{\varepsilon},$$

$$(108) \quad \underline{\Delta}_{H_0^*}(\varphi) = \sum_{i,j=1}^n \langle \varphi, (R^2 g^2 \triangleright {}_i e) (R^1 g^1 \triangleright {}_j e) \rangle {}^i e \otimes {}^j e,$$

$$(109) \quad \underline{\varepsilon}_{H_0^*}(\varphi) = \varphi(\beta),$$

$$(110) \quad \underline{S}_{H_0^*}(\varphi) = v \circ \underline{S},$$

for all $h \in H$ and $\varphi \in H^*$, where $R^{-1} = \sum \overline{R}^1 \otimes \overline{R}^2$, $\{{}_i e\}_{i=\overline{1},n}$ is a basis of H and $\{{}^i e\}_{i=\overline{1},n}$ the corresponding dual basis of H^* . In this particular case we have

$$I_l(H_0^*) = \{\Lambda \in H^* \mid \sum \Lambda(S(\tilde{p}^2) f^1 \overline{R}^1 \triangleright h_{\underline{2}}) S(\tilde{p}^1) f^2 \overline{R}^2 \triangleright h_{\underline{1}} = \Lambda(b)\beta, \text{ for all } h \in H\}.$$

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FACULTY OF MATHEMATICS, UNIVERSITY OF BUCHAREST, RO-70109 BUCHAREST 1, ROMANIA
E-mail address: `dbulacu@al.math.unibuc.ro`

FACULTY OF APPLIED SCIENCES, VRIJE UNIVERSITEIT BRUSSEL, VUB, B-1050 BRUSSELS, BELGIUM
E-mail address: `scaenepe@vub.ac.be`
URL: `http://homepages.vub.ac.be/~scaenepe/`

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, PO-Box 1-764, RO-70700 BUCHAREST, ROMANIA
E-mail address: `florin.panaite@imar.ro`